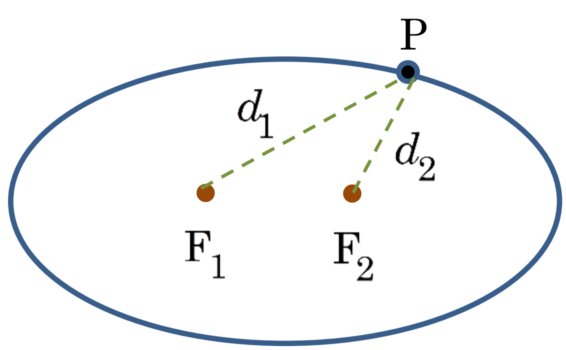
CONIC SECTIONS – THE ELLIPSE

❑ DEFINITION OF THE ELLIPSE

Let F_1 and F_2 be two distinct points in the plane and let C be a positive constant. Then an ***ellipse*** is the set of points in the plane satisfying the following condition: A point is on the ellipse if the sum of its distances to F_1 and F_2 is C, the given constant.



For any point P on the ellipse:

$$d_1 + d_2 = C$$

Each of the points F_1 and F_2 is called a **focus** (plural: **foci**, pronounced $\text{f}\bar{\text{o}}\text{-s}\bar{\text{i}}$).

In the elliptical orbit of the Earth about the Sun, the Sun is one of the two foci of the ellipse.

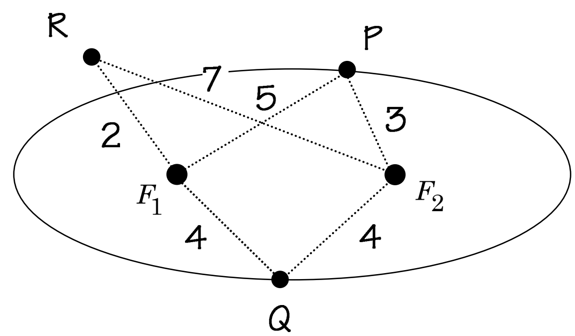
EXAMPLE 1: **In the ellipse graphed below, assume that the constant mentioned in the definition of the ellipse is 8. Explain why points P and Q are on the ellipse and why point R is not on the ellipse.**

Solution: Let's use the notation $d(\text{A}, \text{B})$ to denote the distance between the points A and B. By definition, a point is on the ellipse if the sum of its distances to the two foci is constant — in this example, 8.

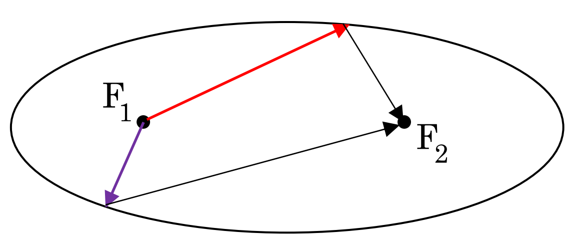
Point P: $d(\text{P}, \text{F}_1) = 5$ and $d(\text{P}, \text{F}_2) = 3$, and so the sum of the distances is 8. Therefore, P is on the ellipse.

Point Q: $d(\text{Q}, \text{F}_1) = 4$ and $d(\text{Q}, \text{F}_2) = 4$, and so the sum of the distances is 8. Thus, Q is on the ellipse.

Point R: $d(\text{R}, \text{F}_1) = 2$ and $d(\text{R}, \text{F}_2) = 7$, which means that the sum of the distances is 9. We conclude that R is not on the ellipse.



The ellipse has a very interesting reflecting property: Any form of energy (for example, sound or light) that emanates from one focus of the ellipse will reflect off the inside of the ellipse right to the other focus. See the two "rays" starting at F_1? Each one leaves the focus, reflects off the inside of the ellipse, and then bounces to F_2, the other focus.



This is the property which makes lithotripsy work: A device which produces sound waves is positioned at one focus, while the patient is positioned so that his kidney stone is positioned at the other. The sound waves that are produced at the first focus reflect off the inside of the machine and are all "focused" at the kidney stone, providing enough sonic energy to break the stone into pieces small enough to pass naturally through the body.

Homework

1. The closer the foci, the more the ellipse looks like a ____.

   [What happens when the foci are the same point? It becomes a circle, with the foci becoming the center of the circle. This brings up the question: Is a circle just a special kind of ellipse? Some books say yes; the foci coincide to become the center. Some say no, a circle is not an ellipse because their definition requires that the foci must be distinct.]

2. Sketch an ellipse where the foci are on the same vertical line.

3. Is it possible for an ellipse to be a function?

4. Let F_1 and F_2 represent the foci of an ellipse. Also suppose that P is a point on the ellipse. Assume further that the distance between P and F_1 is 20 and the distance between P and F_2 is 12. Now let Q be a point on the ellipse such that the distance from Q to F_1 is 30. What is the distance between Q and F_2?

❑ *DERIVING THE FORMULA FOR THE ELLIPSE*

LINK TO A WEBSITE

Ellipses With Center at the Origin

EXAMPLE 2: **Graph the ellipse and state its domain and range:**

$$\frac{x^2}{4}+\frac{y^2}{25} = 1$$

Solution: Let's begin by finding all the intercepts of the ellipse. These points and a couple of others will convince us that the graph is indeed elliptical.

***x*-intercepts:** Set $y = 0$ to get

$$\frac{x^2}{4} = 1 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm\sqrt{4} = \pm 2$$

and so the x-intercepts are (2, 0) and (–2, 0).

***y*-intercepts:** Set $x = 0$ to get

$$\frac{y^2}{25} = 1 \quad \Rightarrow \quad y^2 = 25 \quad \Rightarrow \quad y = \pm\sqrt{25} = \pm 5$$

and thus the y-intercepts are (0, 5) and (0, –5).

additional points: Let $x = 1$. Then

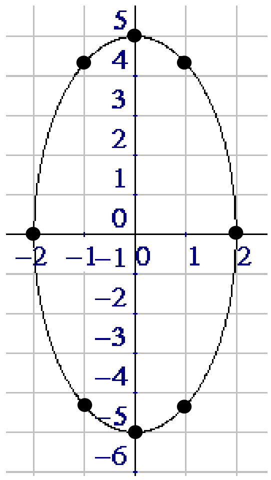
$$\frac{1^2}{4}+\frac{y^2}{25} = 1 \quad \Rightarrow \quad \frac{y^2}{25} = \frac{3}{4} \quad \Rightarrow \quad y^2 = \frac{75}{4}$$

$$\Rightarrow \quad y = \pm\sqrt{\frac{75}{4}} \approx \pm 4.33$$

This yields the points (1, 4.33) and (1, –4.33).

Note that if we let $x = -1$, we obtain the same y-values. Thus, two additional points are (–1, 4.33) and (–1, –4.33).

We now have a total of eight points we can plot. When we do, we get the graph of the ellipse:



Domain = $[-2, 2]$

Range = $[-5, 5]$

Note: It's not a function.

Note: The **center** of the ellipse is the origin.

A final note: Since the domain of the ellipse is $[-2, 2]$, we know that an x-value of 3, for example, would not be allowed in the formula; indeed, there's no point on the ellipse whose x-coordinate is 3. But how can we verify this fact if we don't already know the domain? Let's let $x = 3$ and see what happens:

$$\frac{x^2}{4}+\frac{y^2}{25} = 1 \quad \Rightarrow \quad \frac{3^2}{4}+\frac{y^2}{25} = 1 \quad \Rightarrow \quad \frac{9}{4}+\frac{y^2}{25} = 1$$

$$\Rightarrow \quad \frac{y^2}{25} = 1-\frac{9}{4} \quad \Rightarrow \quad \frac{y^2}{25} = -\frac{5}{4} \quad \Rightarrow \quad y^2 = -\frac{125}{4}$$

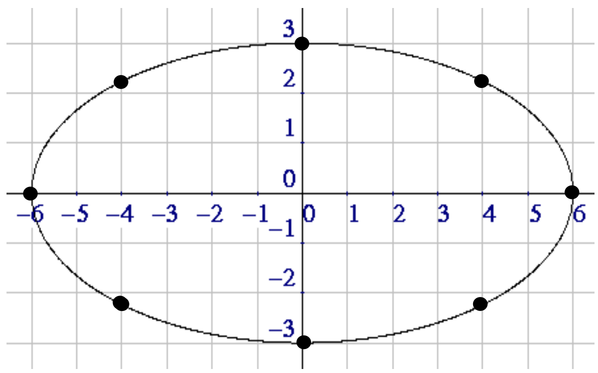
$\Rightarrow y = \pm\sqrt{-\frac{125}{4}}$, which are not real numbers. Since letting $x = 3$ resulted in no solution for y, we conclude that 3 is definitely not in the domain of the ellipse.

EXAMPLE 3: **Graph the ellipse and state its domain and range:**

$$\boldsymbol{\frac{x^2}{36}+\frac{y^2}{9} = 1}$$

Solution: Since this equation is very similar to the previous one, it's likely that its graph is an ellipse with center at the origin. You can calculate the intercepts to be (6, 0), (–6, 0), (0, 3), and (0, –3).

For extra resolution, we can find four more points by letting $x = \pm 4$, and then calculating y to be $\pm\sqrt{5}$. We thus get (4, 2.24), (4, –2.24), (–4, 2.24), and (–4, –2.24). Here's the graph:



Center = (0, 0)

Domain = [–6, 6]

Range = [–3, 3]

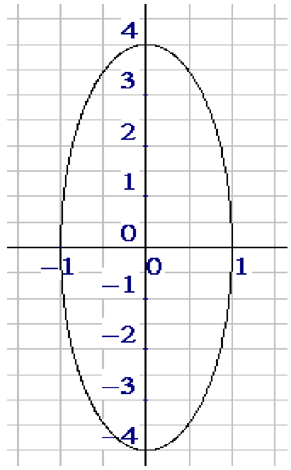
EXAMPLE 4: **Graph the ellipse and state its domain and range:**

$$\mathbf{16x^2 + y^2 = 16}$$

Solution: Is this an ellipse? It doesn't have the same form as the equations in Examples 2 and 3. But we can turn it into that form by dividing each side by 16 (the 16 on the right side of the equation).

$$\frac{16x^2 + y^2}{16} = \frac{16}{16} \Rightarrow x^2 + \frac{y^2}{16} = 1$$

The intercepts are (1, 0), (–1, 0), (0, 4), and (0, –4). That's enough points — we know what the graph looks like:



Center = (0, 0)

Domain = [–1, 1]

Range = [–4, 4]

Homework

5. Prove that $x = 12$ is not in the domain of the ellipse $\frac{x^2}{100} + \frac{y^2}{49} = 1$.

6. Consider the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

   a. Find all four intercepts.
   b. Find two more points on the ellipse by letting $x = 2$.
   c. Find two more points on the ellipse by letting $x = -3$.
   d. Sketch the ellipse.
   e. State the domain and range.
   f. Explain why it's not a function.

Graph each ellipse and state its domain and range:

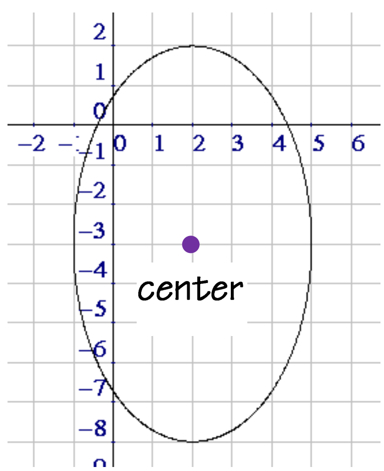
7. $\frac{x^2}{25} + \frac{y^2}{9} = 1$ 8. $x^2 + \frac{y^2}{49} = 1$ 9. $x^2 + 9y^2 = 9$

❑ *ELLIPSES WITH CENTER OFF THE ORIGIN*

EXAMPLE 5: **Graph the ellipse and state its center, domain and range:**

$$\frac{(x-2)^2}{9} + \frac{(y+3)^2}{25} = 1$$

Solution: If the equation were $\frac{x^2}{9} + \frac{y^2}{25} = 1$, we know that it would be an ellipse with center at the origin. The ellipse would extend 3 units right and left (because of the 9), and 5 units up and down (because of the 25), from its center. It's exactly the same idea for the given ellipse, except that the center has been moved 2 units to the right (because of the –2) and 3 units down (because of the +3). [The origin center has been *horizontally translated* to the right and *vertically translated* down.]



Center = (2, –3)

Domain = [–1, 5]

Range = [–8, 2]

EXAMPLE 6: **Graph the ellipse $x^2 + 9y^2 + 4x - 18y + 4 = 0$ and state its center, domain and range.**

Solution: It's not so obvious that the graph is an ellipse. Recall that in the formula for a circle, both variables are squared, and each squared variable has the same coefficient. What makes a formula an ellipse is that both variables are squared, but the coefficients are different. In addition, the two coefficients must have the same sign (this won't make sense until you study hyperbolas).

To find the center of this ellipse, we wish it had been written in the form of Example 5 — but it wasn't, so we'll have to deal with it. Our goal, then, is to take the equation $x^2 + 9y^2 + 4x - 18y + 4 = 0$ and change it into a form where we can recognize its center. As we did with circles, we'll complete the square in both variables; but being that it's an ellipse, we'll make sure that we have a "**1**" on the right side of the equation when we're done.

We take a deep breath and start with the original equation:

$$x^2 + 9y^2 + 4x - 18y + 4 = 0$$

Rearrange the terms and pull the constant to the right:

$$x^2 + 4x + 9y^2 - 18y = -4$$

Factor out the GCF:

$$x^2 + 4x + 9(y^2 - 2y) = -4$$

Complete the squares:

$$x^2 + 4x + \boxed{4} + 9(y^2 - 2y + \boxed{1}) = -4 + \boxed{4} + \boxed{9}$$

Note: We put the magic number of 1 inside the parentheses; therefore, we really added **$9 \cdot 1$** to the left side of the equation. This maneuver, of course, is balanced by adding **9** to the right side of the equation.

Factor the perfect square trinomials and simplify the right side:

$$(x+2)^2 + 9(y-1)^2 = 9$$

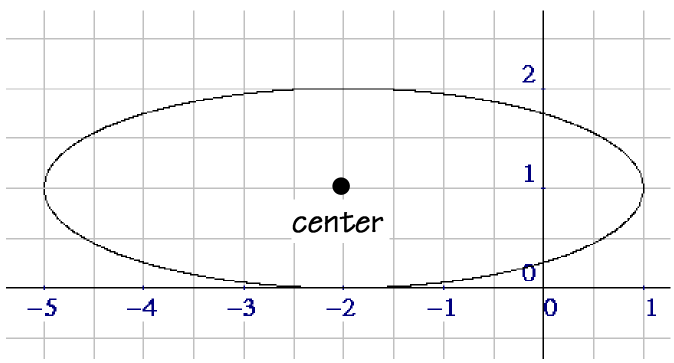
Divide through by 9:

$$\frac{(x+2)^2}{9} + \frac{9(y-1)^2}{9} = \frac{9}{9}$$

which simplifies to

$$\frac{(x+2)^2}{9} + \frac{(y-1)^2}{1} = 1$$

It's now in the proper ellipse form — we see that the center is the point (–2, 1), that it extends 3 units left and right from the center, and 1 unit up and down from the center.



Center = (–2, 1)

Domain = [–5, 1]

Range = [0, 2]

Homework

10. Find the center, the domain, and the range of each ellipse:

a. $\frac{(x-2)^2}{100} + \frac{(y+7)^2}{49} = 1$ b. $\frac{(x+3)^2}{4} + \frac{(y-9)^2}{64} = 1$

c. $(x+3)^2 + \frac{(y-3)^2}{25} = 1$ d. $\frac{(x-5)^2}{81} + y^2 = 1$

Graph each ellipse and state its domain and range:

11. $9x^2 + 4y^2 - 54x + 40y + 37 = 0$
12. $49x^2 + 16y^2 - 294x + 32y - 327 = 0$
13. $4x^2 + y^2 + 24x + 8y + 16 = 0$
14. $x^2 + 16y^2 - 10x + 64y + 73 = 0$
15. $25x^2 + 9y^2 - 50x - 108y + 124 = 0$
16. $x^2 + 4y^2 + 14x - 32y + 97 = 0$
17.** Find the domain of $\frac{5}{9}x^2 + \frac{11}{2}y^2 + \frac{5}{9}x - \frac{22}{3}y + \frac{19}{12} = 0$

Review Problems

18. State the geometric definition of the ellipse.

19. a. Does an ellipse have one or two foci?
    b. Describe where the center of an ellipse is relative to the foci.
    c. The foci of an ellipse are at (2, 10) and (20, 10). What is the center of the ellipse?

20. Let F_1 and F_2 represent the foci of an ellipse. Also suppose that P is a point on the ellipse. Assume further that the distance between P and F_1 is 88 and the distance between P and F_2 is 122. Now let Q be a point on the ellipse such that the distance from Q to F_1 is 115. What is the distance between Q and F_2?

21. Consider the ellipse $\frac{x^2}{9} + \frac{y^2}{100} = 1$.
    a. Prove that $x = 0$ is in the domain.

b. Prove that $x = 4$ is not in the domain.

22. Find all the intercepts, the domain, and the range of $25x^2 + 4y^2 = 100$.

23. Find the center, the domain, and the range of
$25x^2 + 4y^2 - 50x + 56y + 121 = 0$.

24. Do you think that the graph of $\frac{x^2}{123} + \frac{y^2}{123} = 1$ is an ellipse?

25. An elliptical pool table has its only pocket at one of the foci. Where should you place the cue ball so that no matter what direction you hit it, you'll sink it?

26. True/False:

a. An ellipse is the set of points in the plane such that the sum of the distances to two fixed points is a constant.

b. The foci of an ellipse always lie on the same horizontal line.

c. The further apart the foci, the more oblong the ellipse.

d. There's an ellipse that's a function.

e. The ellipse $25x^2 + 4y^2 = 100$ has four intercepts.

f. The range of the ellipse above is $[-5, 5]$.

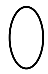
g. $x = 12$ is in the domain of the ellipse $\frac{x^2}{100} + \frac{y^2}{49} = 1$.

h. The center of the ellipse $x^2 + 9y^2 + 4x - 18y + 4 = 0$ is $(-2, 1)$.

i. The graph of $9x^2 - 16y^2 = 144$ is an ellipse.

j. The graph of $10x^2 = y^2 + 100$ is an ellipse.

Solutions

1. Circle **2.**  **3.** No **4.** 2

5. Let $x = 12$ in the ellipse equation and you should end up with y^2 equaling a negative number, resulting in no real solutions.

6. a. (4, 0) (–4, 0) (0, 2) (0, –2)

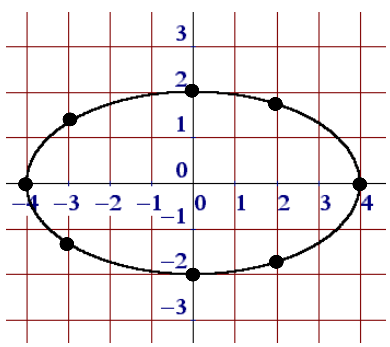
b. Letting $x = 2$ gives

$$\frac{2^2}{16}+\frac{y^2}{4} = 1 \;\Rightarrow\; \frac{1}{4}+\frac{y^2}{4} = 1 \;\Rightarrow\; \frac{y^2}{4} = \frac{3}{4}$$

$\Rightarrow\; y^2 = 3 \;\Rightarrow\; y = \pm\sqrt{3}$, which give us the two approximate points (2, 1.732) and (2, –1.732).

c. Letting $x = -3$ gives the points (–3, 1.333) and (–3, –1.333).

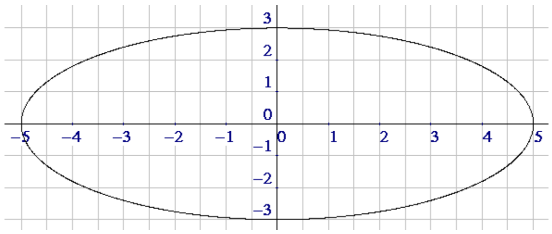
d.



e. Domain = [–4, 4] Range = [–2, 2]

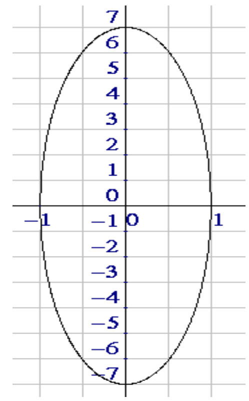
f. Letting the input (x-value) be 0, for example, we get the <u>two</u> outputs (y-values) of 2 and –2. Thus, the ellipse is not a function.

7.



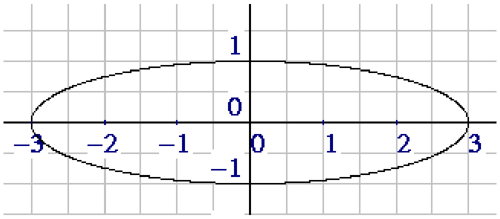
Domain = [–5, 5]
Range = [–3, 3]

8.



Domain = [–1, 1]
Range = [–7, 7]

9.



Domain = [–3, 3]
Range = [–1, 1]

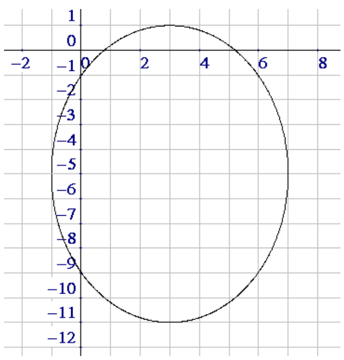
10. a. Center = (2, –7) Domain = [–8, 12] Range = [–14, 0]

b. Center = (–3, 9) Domain = [–5, –1] Range = [1, 17]

c. Center = (–3, 3) Domain = [–4, –2] Range = [–2, 8]

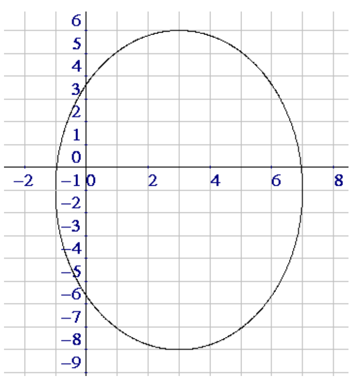
d. Center = (5, 0) Domain = [–4, 14] Range = [–1, 1]

11.



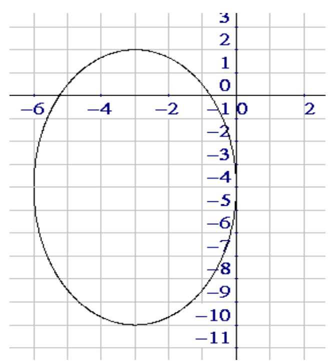
Domain = [–1, 7]
Range = [–11, 1]

12.



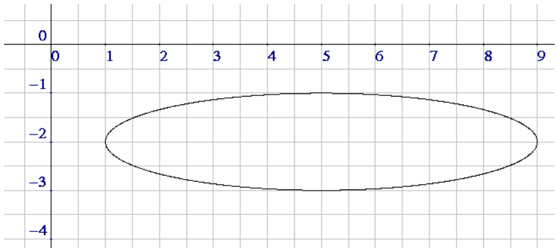
Domain = [–1, 7]
Range = [–8, 6]

13.



Domain = [–6, 0]
Range = [–10, 2]

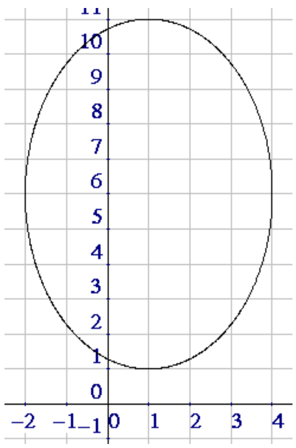
14.



Domain = [1, 9]

Range = [–3, –1]

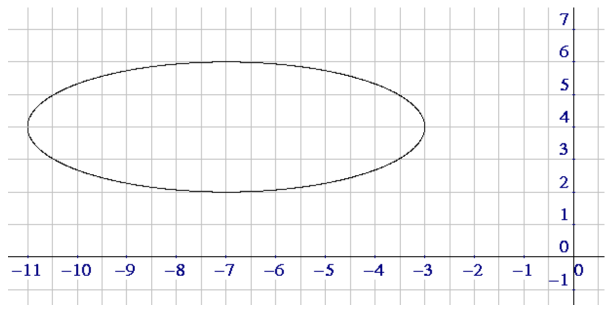
15.



Domain = [–2, 4]

Range = [1, 11]

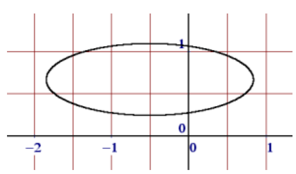
16.



Domain = [–11, –3]

Range = [2, 6]

17. $\left[\dfrac{-5-6\sqrt{5}}{10}, \dfrac{-5+6\sqrt{5}}{10}\right]$



18. An ellipse is the set of points in the plane such that the <u>sum</u> of the distances to two fixed points (the foci) is a constant.

19. a. Most sources say two foci.

b. The center is halfway between the foci, or, the center is the midpoint of the line segment connecting the foci.

c. (11, 10)

20. 95

21. a. Letting $x = 0$ in the ellipse equation results in $y = \pm 10$, which of course are real numbers. Therefore, 0 is in the domain of the ellipse.

b. Letting $x = 4$ in the equation results in $y^2 = -\frac{700}{9}$, which has no solution. Thus, 4 is not in the domain.

22. (±2, 0) (0 ±5); [–2, 2]; [–5, 5]

23. $\frac{(x-1)^2}{4} + \frac{(y+7)^2}{25} = 1$ Center: (1, –7)

Domain: [–1, 3] Range = [–12, –2]

24. Maybe it is, and maybe it isn't.

25. At the other focus.

26. a. T b. F c. T d. F e. T f. T
g. F h. T i. F j. F

Courage and perseverance have a magical talisman, before which difficulties disappear and obstacles vanish into air.

John Quincy Adams